Much of the elementary theory of differential calculus rests on a few simple properties of the families $\mathcal{D}$ and $\mathcal{o}$. These are given in propositions 8.1.6–8.1.14.

**8.1.6. Definition.** A function $L: \mathbb{R} \to \mathbb{R}$ is linear if

$$L(x + y) = L(x) + L(y)$$

and

$$L(cx) = cL(x)$$

for all $x, y, c \in \mathbb{R}$. The family of all linear functions from $\mathbb{R}$ into $\mathbb{R}$ will be denoted by $\mathcal{L}$.

The collection of linear functions from $\mathbb{R}$ into $\mathbb{R}$ is not very impressive, as the next problem shows. When we get to spaces of higher dimension the situation will become more interesting.

**8.1.7. Example.** A function $f: \mathbb{R} \to \mathbb{R}$ is linear if and only if its graph is a (nonvertical) line through the origin.

**Proof.** Problem.

**CAUTION.** Since linear functions must pass through the origin, straight lines are not in general graphs of linear functions.

**8.1.8. Proposition.** Every member of $\mathcal{o}$ belongs to $\mathcal{D}$; so does every member of $\mathcal{L}$. Every member of $\mathcal{D}$ is continuous at 0.

**Proof.** Obvious from the definitions.

**8.1.9. Proposition.** Other than the constant function zero, no linear function belongs to $\mathcal{o}$.

**Proof.** Exercise. (Solution Q.8.1.)

**8.1.10. Proposition.** The family $\mathcal{D}$ is closed under addition and multiplication by constants.

**Proof.** Exercise. (Solution Q.8.2.)

**8.1.11. Proposition.** The family $\mathcal{o}$ is closed under addition and multiplication by constants.

**Proof.** Problem.

The next two propositions say that the composite of a function in $\mathcal{D}$ with one in $\mathcal{o}$ (in either order) ends up in $\mathcal{o}$.

**8.1.12. Proposition.** If $g \in \mathcal{D}$ and $f \in \mathcal{o}$, then $f \circ g \in \mathcal{o}$.

**Proof.** Problem.

**8.1.13. Proposition.** If $g \in \mathcal{o}$ and $f \in \mathcal{D}$, then $f \circ g \in \mathcal{o}$.

**Proof.** Exercise. (Solution Q.8.3.)

**8.1.14. Proposition.** If $\phi, f \in \mathcal{D}$, then $\phi f \in \mathcal{o}$.

**Proof.** Exercise. (Solution Q.8.4.)

**Remark.** The preceding facts can be summarized rather concisely. (Notation: $\mathcal{C}_0$ is the set of all functions in $\mathcal{F}_0$ which are continuous at 0.)

1. $\mathcal{L} \cup \mathcal{o} \subseteq \mathcal{D} \subseteq \mathcal{C}_0$.
2. $\mathcal{L} \cap \mathcal{o} = 0$.
3. $\mathcal{D} + \mathcal{D} \subseteq \mathcal{D}$; $\alpha \mathcal{D} \subseteq \mathcal{D}$.
4. $\mathcal{o} + \mathcal{o} \subseteq \mathcal{o}$; $\alpha \mathcal{o} \subseteq \mathcal{o}$.
5. $\mathcal{o} \circ \mathcal{D} \subseteq \mathcal{o}$.
6. $\mathcal{D} \circ \mathcal{o} \subseteq \mathcal{o}$.
7. $\mathcal{D} \cdot \mathcal{D} \subseteq \mathcal{o}$.