8.4. DIFFERENTIABILITY

8.3.4. Proposition. If \( f, g \in \mathcal{F}_a \), then
\[ \Delta(f + g)_a = \Delta f_a + \Delta g_a.\]

**Proof.** Exercise. (Solution Q.8.8.)

The last two propositions prefigure the fact that differentiation is a linear operator; the next result will lead to *Leibniz’s rule* for differentiating products.

8.3.5. Proposition. If \( \phi, f \in \mathcal{F}_a \), then
\[ \Delta(\phi f)_a = \phi(a) \cdot \Delta f_a + \Delta \phi_a \cdot f(a) + \Delta \phi_a \cdot \Delta f_a.\]

**Proof.** Problem.

Finally, we present a result which prepares the way for the *chain rule*.

8.3.6. Proposition. If \( f \in \mathcal{F}_a \), \( g \in \mathcal{F}_{f(a)} \), and \( g \circ f \in \mathcal{F}_a \), then
\[ \Delta(g \circ f)_a = \Delta g_{f(a)} \cdot \Delta f_a.\]

**Proof.** Exercise. (Solution Q.8.9.)

8.3.7. Proposition. Let \( A \subseteq \mathbb{R} \). A function \( f: A \rightarrow \mathbb{R} \) is continuous at the point \( a \) in \( A \) if and only if \( \Delta f_a \) is continuous at 0.

**Proof.** Problem.

8.3.8. Proposition. If \( f: U \rightarrow U_1 \) is a bijection between subsets of \( \mathbb{R} \), then for each \( a \) in \( U \) the function \( \Delta f_a: U - a \rightarrow U_1 - f(a) \) is invertible and
\[ (\Delta f_a)^{-1} = \Delta(f^{-1})_{f(a)}.\]

**Proof.** Problem.

8.4. DIFFERENTIABILITY

We now have developed enough machinery to talk sensibly about differentiating real valued functions.

8.4.1. Definition. Let \( f \in \mathcal{F}_a \). We say that \( f \) is DIFFERENTIABLE AT \( a \) if there exists a linear function which is tangent at 0 to \( \Delta f_a \). If such a function exists, it is called the DIFFERENTIAL of \( f \) at \( a \) and is denoted by \( df_a \). (Don’t be put off by the slightly complicated notation; \( df_a \) is just a member of \( \mathcal{L} \) satisfying \( df_a \simeq \Delta f_a \).) We denote by \( \mathcal{D}_a \) the family of all functions in \( \mathcal{F}_a \) which are differentiable at \( a \).

The next proposition justifies the use of the definite article which modifies “differential” in the preceding paragraph.

8.4.2. Proposition. Let \( f \in \mathcal{F}_a \). If \( f \) is differentiable at \( a \), then its differential is unique. (That is, there is at most one linear map tangent at 0 to \( \Delta f_a \).)

**Proof.** Proposition 8.2.5. \( \square \)

8.4.3. Example. It is instructive to examine the relationship between the differential of \( f \) at \( a \), which we defined in 8.4.1, and the derivative of \( f \) at \( a \) as defined in beginning calculus. For \( f \in \mathcal{F}_a \) to be differentiable at \( a \) it is necessary that there be a linear function \( T: \mathbb{R} \rightarrow \mathbb{R} \) which is tangent at 0 to \( \Delta f_a \). According to 8.1.7 there must exist a constant \( c \) such that \( Tx = cx \) for all \( x \) in \( \mathbb{R} \). For \( T \) to be tangent to \( \Delta f_a \), it must be the case that
\[ \Delta f_a - T \in 0; \]
that is,
\[ \lim_{h \to 0} \frac{\Delta f_a(h) - ch}{h} = 0.\]